Int. J. Solids Structures Vol. 29, No. 21, pp. 2597–2611, 1992
Printed in Great Britain.
0020-7683/92 $5.00 + .00
Pergamon Press Ltd

# THEORY OF DYNAMIC BENDING OF THIN ELASTIC HIGH NONHOMOGENEOUS PLATES†

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(Received 6 February 1992)

**Abstract**—Primary bending motions of thin plates with an arbitrary hierarchy of layers are considered. Large differences in thicknesses, Young's moduli and sound velocities are taken into account in view of additional small parameters in 3-D equations of the theory of elasticity. Based on the asymptotical analysis the general approximate 2-D equations are deduced first. Then they are simplified and classified depending on asymptotical powers of the mentioned differences and locations of layers.

## 1. INTRODUCTION

Optimum design makes high demands on thin-walled structures working under unfavorable multiform influences. Such demands can often be satisfied by layered structures with a wide variety of layers.

Methods based on the engineering approaches and physical hypotheses [for example, Bolotin and Novitchkov (1980) and Karmishin *et al.* (1990)] and methods of asymptotical integration of 3-D equations of the theory of elasticity (Gol'denveiser, 1962; Gussein-Zade, 1968, 1970; Simonov, 1989) are mainly used in the construction of 2-D mathematical models of such structures. The first of them possess a simplicity and a possibility of the use of the variation principles. The second of these often produce more complex computations and results. But they enable us to establish the limits of justification of the hypotheses, to determine all the stress tensor components. This last point is especially important for high nonhomogeneous structures as, for weak layers, the stress components tend to be comparable to the asymptotic powers and the bending stresses are not dominant. Besides, the minor stresses are necessary for a statement of more complex mixed local problems, for example, about the thin longitudinal inclusion (crack) in a plate.

A spectrum-analysis of the 3-D problem for a nonhomogeneous plate shows that the solution may be represented approximately in view of the internal integral and boundary layer effects (Vorovich *et al.*, 1975). Some results concerning the plates with alternately high and low modulus layers are given by Bolotin and Novichkov (1980) and Gussein-Zade (1968, 1970).

This paper uses an asymptotic integration of 3-D equations for thin isotropic elastic high nonhomogeneous layered plates. Besides the usual small parameter $\varepsilon$ in the theory of shells and plates other small parameters such as the ratios of Young's moduli, thicknesses and sound velocities are considered. By the two last assumptions as well as various Poisson's ratios this work is fundamentally different from previous research. In particular, such an approach enables us to study structures consisting of very thin supporting high-modulus layers and of internal low-modulus aggregates. At $\varepsilon \to 0$ the stated ratios are assumed to degenerate according to the given power laws.

Exact 2-D equations for the expansion coefficients of unknown functions in power sets with respect to transverse coordinates are transformed to the recurrent form. It is convenient for the development of the theories providing a range of accuracies. The first iteration gives 2-D equations with a wide choice of parameters. Then the variants produced by the fundamental simplifications of the general model are classified. Thus, the improved classical theory is derived. It is of great significance because the analogous theory has only been examined (Gussein-Zade, 1966) for the following states: (1) static; (2) identical Poisson's

† Transmitted by Professor Dr V. V. Bolotin.

ratios; (3) symmetric sets of laminae in plates. The similarity with Bolotin's theory is observed for structures with two alternating layers. A class of plates is described by principal equations with the same complexity as in the classical theory whilst retaining strong nonhomogeneity. We also investigate how the usual physical assumptions for thin plates lose (in turn) their force at the increasing nonhomogeneity. A discussion of the lateral boundary conditions is found at the end of this paper.

## 2. PHYSICAL ASSUMPTIONS AND MATHEMATICAL PROBLEM

Let a plate consist of $N$ isotropic homogeneous elastic layers with full contact and occupy a region $-\infty < x, y < \infty$, $z_1 < z < z_{N+1} \equiv z_0$ in Cartesian coordinates. Let $z_i$, $h_i = z_{i+1} - z_i$, $h = \frac{1}{2}(z_0 - z_1)$, $\rho_i$, $E_i$, $\nu_i$, $i \in I = (i\colon 1 \leqslant i \leqslant N)$ be the coordinates of faces, thicknesses of the laminae, semi-thickness of the plate, densities, Young's modulus and Poisson's ratios. Until time $t < 0$ the plate is at rest, then it is moving at $t \geqslant 0$ under the influence of loads $\sigma^m_{\alpha z *}(x, y, t)$ on its faces $z = z_1, z_0$, where $\sigma^m_{\alpha z *}$ are regular and sufficiently weakly varying functions (everywhere $\alpha = x, y, z$, $\beta = x, y$, $m = 1, N$, $\alpha_{\beta\beta} = \sigma_\beta$). We use this physical situation for the construction of 2-D asymptotically exact equations for the dynamic bending–stretching–shear description of the high contrast layered plates. The latter are defined by means of the following asymptotic equalities:

$$e_i = \varepsilon^{p_i} E_{\max}/E_i = O(1), \quad C_i = \varepsilon^{q_i} c_i^2/c_{\min}^2 = O(1), \quad H_i = h_i/h = O(\varepsilon^{r_i}), \quad \varepsilon \Rightarrow 0,$$

$$E_{\max} = \max_{i \in I} E_i \equiv \max E_i, \quad c_{\min} = \min c_i, \quad c_i^2 = E_i/\rho_i.$$

Here $\varepsilon = h/l$ is a familiar small parameter, $l$ is a scale of the process in the longitudinal directions, $p_i$, $q_i$, $r_i$ are known non-negative indices of contrasts of laminae. Let us also introduce non-dimensional functions and independent variables

$$E_{\max} V_\alpha = h\sigma_* v_\alpha, \quad \sigma_{\alpha\beta} = \sigma_* \tau_{\alpha\beta}, \quad t = t_0 \tau, \quad z = h\zeta, \quad x, y = l\cdot(\xi, \eta), \quad t_0 = l \cdot c_{\min}^{-1} \varepsilon^{(\gamma-1)/2}. \tag{1}$$

Here $V_\alpha, \sigma_{\alpha\beta}$ are the displacement vector and stress tensor components, $\sigma_*$ is a peak of loads, $t_0$ denotes time and $\gamma$, evaluated below, is its index. The values $\nu_i$, $1 - 2\nu_i$ are not assumed to be very small.

All functions are represented with respect to the transverse coordinate,

$$v_\alpha = \sum_{k=0} \zeta^k v_{\alpha k}, \quad V_\alpha = \sum_{k=0} z^k V_{\alpha k}, \tag{2}$$

and substitute (1), (2) in the original dynamic boundary problem of the 3-D linear elastic theory (the numbers of terms in sums of (2) will vary according to the given asymptotic accuracy). Then the following system of equalities can be derived for the coefficients of (2), which are functions of $\xi$, $\eta$, $\tau$:

$$k(k-1)v_{\beta k} = 2(1+\nu)C_i \varepsilon^{Q_i} \frac{\partial^2 v_{\beta k-2}}{\partial \tau^2} - \frac{k-1}{1-2\nu}\varepsilon \frac{\partial v_{\zeta k-1}}{\partial \beta} - \frac{\varepsilon^2}{1-2\nu}\frac{\partial}{\partial \beta} \operatorname{div} \mathbf{v}_{k-2} - \varepsilon^2 \Delta v_{\beta k-2},$$

$$k(k-1)v_{\zeta k} = \frac{C_i \varepsilon^{Q_i}}{\lambda + 2\mu}\frac{\partial^2 v_{\zeta k-2}}{\partial \tau^2} - \frac{k-1}{2(1-\nu)}\varepsilon \operatorname{div} \mathbf{v}_{k-1} - \frac{1-2\nu}{2(1-\nu)}\varepsilon^2 \Delta v_{\zeta k-2}, \quad (k = 2, 3, \ldots) \tag{3a}$$

$$\tau_{\zeta k} = e_i \varepsilon^{p_i}\{(k+1)(\lambda + 2\mu)v_{\zeta k+1} + \varepsilon\lambda \operatorname{div} \mathbf{v}_k\}; \quad \mathbf{v} = (v_\zeta, v_\eta),$$

$$\tau_{\xi k} = \alpha_i \varepsilon^{p_i}\left(\frac{\partial v_{\xi k}}{\partial \xi} + \nu \frac{\partial v_{\eta k}}{\partial \eta}\right) + \frac{\nu \tau_{\zeta k}}{1-\nu}, \quad \tau_{\xi\eta k} = \mu e_i \varepsilon^{p_i}\left(\frac{\partial v_{\xi k}}{\partial \eta} + \frac{\partial v_{\eta k}}{\partial \xi}\right),$$

$$\tau_{\beta\zeta k} = \mu e_i \varepsilon^{p_i}\{(k+1)v_{\beta k+1} + \varepsilon\, \partial v_{\zeta k}/\partial\beta\}, \quad (k = 1, 2, \ldots, \xi \leftrightarrows \eta, \quad \beta = \xi, \eta), \tag{3b}$$

$$\tau^m_{x\zeta 0} = \tau^m_{x\zeta *} - \sum_{k=1} \zeta^k_* \tau^m_{x\zeta k}; \quad \zeta_* = \zeta_1, \quad m = 1; \quad \zeta_* = \zeta_{N+1}, \quad m = N,$$

$$\tau^{i+1}_{x\zeta 0} = \tau^i_{x\zeta 0} + \sum_{k=1} \zeta^k_{i+1}(\tau^i_{x\zeta k} - \tau^{i+1}_{x\zeta k}), \quad (i = 1, 2, \ldots, N-1), \tag{3c}$$

$$v^{i+1}_{x0} = v^i_{x0} + \sum_{k=1} \zeta^k_{i+1}(v^i_{xk} - v^{i+1}_{xk}), \quad v_{xk} = \partial v_{xk}/\partial t = 0, \quad t = 0, \quad (k = 0, 1, \ldots),$$

$$\operatorname{div} \mathbf{v} = \partial v_\xi/\partial\xi + \partial v_\eta/\partial\eta, \quad \Delta = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2, \quad \mu = \frac{1}{2(1+\nu)},$$

$$\lambda = \frac{\nu}{(1+\nu)(1-2\nu)}, \quad \lambda + 2\mu = \frac{1-\nu}{(1+\nu)(1-2\nu)}, \quad \alpha_i = \frac{e_i}{1+\nu_i^2}. \tag{3d}$$

Index $i$ is omitted in obvious cases. The system of equations (3) is not ready to be realized as a recurrent process. Only the first two equations are in the recurrent form. Therefore, the next step consists of the reduction of relations (3b) to the equivalent recurrent form. We write the final result only for the components $\tau_{x\zeta}$ because the other components are determined afterwards from Hooke's law:

$$\tau_{\beta\zeta 1} = e_i C_i \varepsilon^{s_i} \frac{\partial^2 v_{\beta 0}}{\partial\tau^2} - \alpha_i \varepsilon^{2+p_i} \Lambda_{i\beta} \mathbf{v}_0 - \frac{\nu\varepsilon}{1-\nu}\frac{\partial \tau_{\zeta 0}}{\partial\beta},$$

$$k\tau_{\beta\zeta k} = e_i C_i \varepsilon^{s_i} \frac{\partial^2 v_{\beta k-1}}{\partial\tau^2} + \frac{\alpha_i}{k-1}\frac{\partial\Delta}{\partial\beta} v_{\zeta k-2}\varepsilon^{3+p_i} - \frac{2\varepsilon^2}{(k-1)(1-\nu)}\Lambda_{i\beta}\tau_{\beta\zeta k-2}$$

$$- \frac{\varepsilon\nu}{1-\nu}\frac{\partial\tau_{\zeta k-1}}{\partial\beta}, \quad S_i = Q_i + p_i = 4 + p_i + q_i - \gamma \quad (k = 2, 3, \ldots); \quad \boldsymbol{\tau} = (\tau_{\xi\zeta}, \tau_{\eta\zeta}) \tag{4a}$$

$$k\tau_{\zeta k} = e_i C_i \varepsilon^{s_i} \frac{\partial^2 v_{\zeta k-1}}{\partial\tau^2} - \varepsilon \operatorname{div} \boldsymbol{\tau}_{k-1} \quad (k = 1, 2, \ldots)$$

$$\tau^i_{\zeta 1} - e_i C_i \varepsilon^{s_i} \frac{\partial^2 v^i_{\zeta 0}}{\partial\tau^2} = \varepsilon \operatorname{div}\left[\sum_{k=1}\left\{\sum_{j=1}^{i-1}(\zeta^k_j - \zeta^k_{j+1})\boldsymbol{\tau}^j_k + \zeta^k_i \boldsymbol{\tau}^i_k\right\} - \boldsymbol{\tau}^1_*\right]$$

$$= \varepsilon \operatorname{div}\left[\sum_{k=1}\left\{\sum_{j=i+1}^{N}(\zeta^k_{j+1} - \zeta^k_j)\boldsymbol{\tau}^j_k + \zeta^k_{i+1}\boldsymbol{\tau}^i_k\right\} - \boldsymbol{\tau}^N_*\right]$$

$$= \tfrac{1}{2}\varepsilon \operatorname{div}\left[\sum_{k=1}\sum_{j\in I}\{\zeta^k_j \operatorname{sgn}_-(i-j) - \zeta^k_{j+1}\operatorname{sgn}_+(i-j)\}\boldsymbol{\tau}^j_k - \boldsymbol{\tau}^1_* - \boldsymbol{\tau}^N_*\right],$$

$$2\tau_{\zeta 2} = e_i C_i \varepsilon^{s_i}\frac{\partial^2}{\partial\tau^2}(v_{\zeta 1} - \varepsilon \operatorname{div} \mathbf{v}_0) + \frac{\nu\varepsilon^2}{1-\nu}\Delta\tau_{\zeta 0} + \alpha_i \varepsilon^{3+p_i}\Delta \operatorname{div} \mathbf{v}_0,$$

$$k(k-1)\tau_{\zeta k} = e_i C_i \varepsilon^{s_i}\frac{\partial^2}{\partial\tau^2}[(k-1)v_{\zeta k-1} - \varepsilon \operatorname{div} \mathbf{v}_{k-2}]$$

$$- \frac{\alpha_i \varepsilon^{4+p_i}}{k-2}\Delta^2 v_{\zeta k-3} + \frac{\nu\varepsilon^2}{1-\nu}\Delta\tau_{\zeta k-2} + \frac{2\varepsilon^3}{(1-\nu)(k-2)}\Delta \operatorname{div} \boldsymbol{\tau}_{k-3} \quad (k = 3, 4, \ldots), \tag{4b}$$

$$\tau^i_{x\zeta 0} = \tau^1_{x\zeta *} + \sum_{k=1}\left\{\sum_{j=1}^{i-1}(\zeta^k_{j+1} - \zeta^k_j)\tau^j_{x\zeta k} - \zeta^k_i \tau^i_{x\zeta k}\right\}$$

$$= \tau^N_{x\zeta *} + \sum_{k=1}\left\{\sum_{j=i+1}^{N}(\zeta^k_j - \zeta^k_{j+1})\tau^j_{x\zeta k} - \zeta^k_{i+1}\tau^i_{x\zeta k}\right\}$$

$$= \tfrac{1}{2}(\tau^N_{x\zeta *} + \tau^1_{x\zeta *}) - \frac{1}{2}\sum_{k=1}\sum_{j\in I}\{\zeta^k_j \operatorname{sgn}_-(i-j) - \zeta^k_{j+1}\operatorname{sgn}_+(i-j)\}\tau^j_{x\zeta k},$$

$$v_{\beta 1} = \varepsilon^{-p_i}\frac{\tau_{\beta\zeta 0}}{\mu e_i} - \varepsilon\frac{\partial v_{\zeta 0}}{\partial \beta}, \quad v_{\zeta 1} = \frac{\varepsilon^{-p_i}\tau_{\zeta 0}}{e_i(\lambda+2\mu)} - \frac{\varepsilon\nu}{1-\nu}\operatorname{div}\mathbf{v}_0,$$

$$2\Lambda_{i\beta}\mathbf{v}_0 = (1+\nu_i)\frac{\partial}{\partial\beta}\operatorname{div}\mathbf{v}_0 + (1-\nu_i)\Delta v_{\beta 0},$$

$$\operatorname{sgn}_- x = \begin{cases} -1, & x < 0 \\ 1, & x \geqslant 0 \end{cases}, \quad \operatorname{sgn}_+ x = \begin{cases} -1, & x \leqslant 0 \\ 1, & x > 0 \end{cases}. \tag{4c}$$

Here we also show the transformed equations (3c), where the functions $\tau_{\zeta 0}^{i+1}$ are excluded, relations for $\tau_{\alpha\zeta}$ which automatically satisfy the conditions of continuity and for $v_{\alpha 1}$. The system (3a), (3d), (4) is equivalent to the original equations (3), since it is deduced using the nondegenerate algebraic transformations. It has a recurrent form. All the coefficients with the numbers $k$ are expressed through the factors of the numbers $k-1$, $k-2, \ldots$ in the "horizontal" relations (3a), (4a), (4b), (4d). The "vertical" relations (3b), (4c) being equivalent to the various boundary conditions will be given $6N$ infinite length equations for $6N$ functions $\tau_{\alpha\zeta 0}^i$, $v_{\alpha 0}^i$ after substitutions of respective expressions from the recurrent relations. The equalities (3d) and (4c) contain $3N+1$ and $3N-1$ independent correlations, respectively. The triple expression (4c) for $\tau_{\alpha\zeta 0}$ through $\tau_{\alpha\zeta k}$ ($k \geqslant 1$) was deduced by means of their step-by-step exception from the conditions (3c). If this exception has been begun from the lower ($z = z_1$) or upper ($z = z_0$) face of the plate then we have derived the first or second expression (4c). The third equality is the superposition of the first two. The function $\tau_{\zeta 1}$ in (4b) also takes the triple representation for this reason. Further, some form will be chosen from a convenience. It is important for the future that the main terms in the mentioned expressions can be eliminated by virtue of the existence of the other equalities.

The quantities $\tau_{\alpha\zeta 0}^i$ can be excluded from (4c). Then three general equations represent our problem:

$$\sum_{k=1}\sum_{j\in I}(\zeta_{j+1}^k - \zeta_j^k)\tau_{\alpha\zeta k}^j = \tau_{\alpha\zeta *}^N - \tau_{\alpha\zeta *}^1. \tag{5}$$

Properly speaking, eqns (5) are the projections of the vector dynamic force balance equation on the three directions.

Thus the complete system of equations for all the factors in eqns (2) has been derived. We can expect the asymptotic orders of the smallest of the functions $v_{\alpha k}$, $\tau_{\alpha\beta k}$ to increase as $k$ increases. Some limitations on $\gamma, p_i, \ldots$ may be required. Then the iteration process will be finite for a given accuracy.

## 3. GENERAL 2-D EQUATIONS

In this paper our consideration will be restricted by the first iteration. We construct 2-D approximate equations with the relative error $O(\varepsilon^2)$, i.e. all the terms $O(\varepsilon^k)$, $k \geqslant 2$ in comparison to $O(1)$ are omitted from the computations. This degree of accuracy is similar to that of the ordinary Kirchhoff type theories of homogeneous and layered plates. Naturally, the accuracy achievable by the various lateral boundary conditions for high nonhomogeneous plates of limited size is too complicated a question to answer, in general, and it requires some cumbersome analysis. We shall only discuss this briefly at the end of the paper.

Now we limit the index values $\gamma, p_i, q_i$, our assumption $\varepsilon \ll 1$ should be true and some impractical models should be immediately discounted. So, first we can conclude that $Q_i > 0$, $\gamma < 4$. If $Q_{ik} \leqslant 0$ then $v_{\alpha k} = O(\partial^2 v_{\alpha k-2}/\partial\tau^2)$ for some $i \in I$ from (3a) and all the terms of expansion (2) hold for a given accuracy. The physical meaning of this condition is that the characteristic time $t_0$ must be greater than the stress wave of distance $h_i$. At $Q_i \leqslant 0$ the volume vibration of the low velocity and relatively thick laminae becomes essential. However, the elimination of the short wave of bending vibrations yields the stronger restriction $\gamma \leqslant 2$, as will be shown below.

The indexes $p_i, q_i$ seem to be restricted by practical considerations. So, the ratio of module $E$ of steel to the one of weakly porous resin or foam plastic ranges between $10^2$ and $10^3$. Assuming $\varepsilon < 0.1$ we can conclude that the condition $p_i \leqslant 3$ is almost sufficient. As for steel and air, the ratio $c_\alpha^2/c_g^2 \approx 0.003$ so the inequality $q_i \leqslant 2$ is acceptable. Thus we shall take into account the following estimations for indices

$$0 \leqslant p_1 \leqslant 3, \quad 0 \leqslant q_i \leqslant 2, \quad \gamma \leqslant 2. \tag{6}$$

The values $r_i$ are not restricted. Sometimes a group of the thin neighboring laminae with near characteristics will be united in one nonhomogeneous ply with a value $r$. A number $N_*$ of such kind of groups of laminae must be a few ($N_* \ll \varepsilon^{-1}$) according to the meaning of account.

We propose the following asymptotic evaluations for unknown functions under conditions (6)

$$V_z^i = W(x, y, t) + O(\varepsilon^n W), \quad V_{\beta 0}^i = O(\varepsilon^k W), \quad \gamma \leqslant 2 \quad (k, n \geqslant 1). \tag{7}$$

Physically, the relations (7) indicate a nondeflection of the normal across the whole section and an asymptotic tangential displacement is small in comparison with the normal displacement. In fact, they are necessary in order to begin the recurrent process and can be verified *a posteriori*. An example of such an approach is the reflection of the inverse character of the problem of the construction of 2-D equations and it is correct because of the accuracy of the original problem, i.e. the solution exists and is unique.

Owing to (3), (4) some terms in the equations of the exact mathematical model can be evaluated leaving only the main terms in the relations. Omitting the cumbersome computations we write a complete system of approximate equations for the $2N+1$ functions $W$ and $V_{\beta 0}^i$ and the relations for all the stresses and complete displacements $V_\alpha^i$:

$$LW \equiv (D\Delta^2 + 2h\rho\, \partial^2/\partial t^2) W = A_i' \Delta V_i + q_z^- + h \operatorname{div} \mathbf{q}^+, \tag{8a}$$

$$A_i \Lambda_{i\beta} \mathrm{V}_0^i = -q_{\beta z}^- ; \; q_{\alpha z}^\pm = \sigma_{\alpha z *}^N \pm \sigma_{\alpha z *}^1, \quad \mathbf{q} = (q_{xz}, q_{yz}), \tag{8b}$$

$$V_{\beta 0}^i = V_{\beta *} + b_i \sigma_{\beta z *}^m + \sum_{j \in I} \Gamma_{ij}^m \Lambda_{j\beta} \mathrm{V}_0^j - G_i^m \Delta W_{,\beta}, \tag{8c}$$

$$V_\beta^i = V_{\beta 0}^i + z(\mu_i^{-1} \sigma_{\beta z 0}^i - W_{,\beta}) + O(\varepsilon^2 V_\beta^i), \quad V_z^i = W + O(\varepsilon^2 W), \quad p_i - r_i \leqslant 2, \tag{8d}$$

$$V_z^i = W + W^i(x, y, t) + z\sigma_{z0}^i(\lambda_i + 2\mu_i) + O(\varepsilon^2 V_z^i), \quad 2 < p_i - r_i \leqslant 3, \tag{8e}$$

$$\sigma_{\beta z}^i = \sum_{k=0}^{2} z^k \sigma_{\beta z k}^i, \quad \sigma_z^i = \sum_{k=0}^{3} z^k \sigma_{zk}^i, \quad \sigma_x^i = \alpha_i (V_{x,x}^i + \nu_i V_{y,y}^i) + \frac{\nu_i \sigma_z^i}{1+\nu_i}, \quad (x \Leftrightarrow y),$$

$$\sigma_{xy}^i = \mu_i \operatorname{div}^* \mathrm{V}^i, \quad \sigma_{z0}^i = \sigma_{z*}^m + \sum_{k=1}^{3} \left[ \sum_{j \in J_m} (z_{j+1}^k - z_j^k) \sigma_{zk}^j \quad \operatorname{sgn}(i-j) - \sigma_{zk}^i z_{i_m}^k \right],$$

$$\sigma_{z1}^i = \rho_i W_{,tt} + \sum_{j \in J_m} A_j [z_{j+1/2} \Delta^2 W - \Delta V_j] \operatorname{sgn}(i-j) + \alpha_i z_{i_m} (z_{i_m} \Delta^2 W - \Delta V_i) - \operatorname{div} \boldsymbol{\sigma}_*^m,$$

$$\sigma_{z2}^i = \frac{\alpha_i}{2} \Delta V_i, \quad \sigma_{z3}^i = -\frac{\alpha_i}{6} \Delta^2 W, \quad \sigma_{\beta z 1}^i = -\alpha_i \Lambda_{i\beta} V_{\beta 0}^i, \quad \sigma_{\beta z 2} = \frac{\alpha_i}{2} \Delta W_{,\beta},$$

$$\sigma_{\beta z 0}^i = \sigma_{\beta z *}^m - \sum_{j \in J_m} A_j \operatorname{sgn}(i-j) \Lambda_{j\beta} V_{\beta 0}^j + z_{i_m} \alpha_i \Lambda_{i\beta} V_{\beta 0}^i + F_i^m \Delta W_{,\beta}, \quad D = \frac{\alpha_i}{2} (z_{i+1}^3 - z_i^3),$$

$$\rho = \frac{\rho_i h_i}{2h}, \quad A_i' = \frac{A_i}{2} \sum_{j \in I} h_j \operatorname{sgn}(i-j), \quad z_{j+1/2} = \tfrac{1}{2}(z_j + z_{j+1}),$$

$$b_i = \sum_{k\in K} B_k^{-1}\,\mathrm{sgn}\,(i-i_*) - \mu_i^{-1} z_{i+\theta}, \quad \Gamma_{ij}^m = A_i G_{ij}^m, \quad G_i^m = \sum_{j\in I} A_j z_{j+1/2} G_{ij}^m,$$

$$G_{ij}^m = \sum_{k\in K} B_k^{-1} H_{kj}^m\,\mathrm{sgn}\,(i_*-i) + \mu_i^{-1} z_{i+\theta} H_{ij}^m, \quad V_i = \mathrm{div}\, V_{\beta 0}^i, \quad W_{,x} = \frac{\partial W}{\partial x},$$

$$K = \{k : k = i_*+1,\ldots,i-1, i > i_*+1;\ k = i_*-1,\ldots,i+1, i < i_*-1;\ \phi, i = i_*, i_*\pm 1\},$$

$$H_{kj}^1 \equiv H_+(k-j) = \begin{cases} 1, & k > j \\ 0, & k \leqslant j \end{cases}, \quad H_{kj}^N = -H_+(j-k); \quad A_k = \alpha_k h_k, \quad B_k = \mu_k h_k^{-1}; \quad m = 1, N,$$

$$F_i^m = \sum_{j\in J_m} A_j z_{j+1/2}\,\mathrm{sgn}\,(i-j) - \tfrac{1}{2}\alpha_i z_{i_m}^2, \quad i_m : i_1 = i, \quad i_N = i+1, \quad J_1 = \{j : 1 \leqslant j < i\},$$

$$J_N = \{j : i < j \leqslant N\}; \quad \theta = H_+(i_*-i), \quad \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2; \quad \mathrm{div}\,\mathbf{V} = V_{x,x} + V_{y,y};$$

$$\mathrm{div}^*\,\mathbf{V} = V_{x,y} + V_{y,x}; \quad \boldsymbol{\sigma}_* = (\sigma_{xz}, \sigma_{yz})_*, \quad \alpha_i = E_i/(1-\nu_i^2),$$

$$\lambda_i + 2\mu_i = (1-\nu_i)E_i/(1-2\nu_i)(1+\nu_i), \quad \mu_i = E_i/2(1+\nu_i). \tag{8f}$$

Here and below a summation with respect to alternate indices is performed, $V_{\beta*} \equiv V_{\beta i_*}$, $i_*$ is *a number of the reference ply* which is defined as *a rigid ply* ($p_i = 0$) *having maximum thickness*. We can add the third variant to two equivalent forms of eqns (8c) at $m = 1$ and $m = N$ using the change $\sigma_{\beta z*}^m \Rightarrow \frac{1}{2}q_{\beta z}^+$, $H_{ij}^m \Rightarrow \frac{1}{2}\,\mathrm{sgn}\,(i-j)$ and summation of formulae in (8f) at $m = 1$ and $m = N$ gives the new expressions for $\sigma_{zz0}^i$ and $\sigma_{z1}^i$ too.

By the way, we have proved that the equations of the in-plane problem (8b), (8c) are *quasistatic*. The location of a middle plane or center of bending is defined from the condition that a factor of the former main term $\Delta W_{,\beta}$ in eqn (6) is equal to zero:

$$\sum_{i\in I} \alpha_i (z_{i+1}^2 - z_i^2) = 0, \tag{9}$$

and the same fact has been derived more than once from another concept. Using (6), (7) we can establish that in the range $0 \leqslant p_i - r_i \leqslant 2$, $0 \leqslant q_i \leqslant 2$, $i \in I$ the eliminated terms in the equation for $W$ (8a) are $O(\varepsilon^2)$ and $O(\varepsilon)$ in the "membrane equations" (8b), (8c). If $2 \leqslant p_i - r_i \leqslant 3$ for some soft laminae $i \in I_s$ then the corrections $W^i = O(\varepsilon^n W)$, $1 \leqslant n < 2$ in (9) due to the influence of the normal transversal deformation need to be introduced by using the perturbation method (otherwise $W^i \equiv 0$). Then the solution takes the form (8e), where $W$, $V_{\beta 0}^i$, $\sigma_{z0}^i$ are the main solutions, $W^i$ are the corrections expressed at $1 \neq i_*$ through the reference correcting function $W_*$ as follows:

$$W^i = W_* + \sum_{k\in K_s} \frac{h_k}{\lambda_k + 2\mu_k}\,\mathrm{sgn}\,(i-i_*)\sigma_{z0}^k - \frac{z_{i+\theta}}{\lambda_i + 2\mu_i}\sigma_{z0}^i, \tag{10}$$

where $K_s = K \cap I_s$ and the symbol $\cap$ denotes the intersection of sets. By means of the substitution of (8e), (10) in (8f) and then (8f) in (5) at $\alpha = z$ we can obtain for $W_*$ a usual Germen–Lagrange equation with an effective load depending on the result of the first iteration. The remaining functions $V_{\beta 0}^i$, $i \in I_*$ are determined from the continuity conditions (3d) and are not entranced in the above expressions. The complete tangential displacements are given by expressions (8c) with the additional term $-z\,\mathrm{d}W^i/\mathrm{d}\beta$.

Passing to the limits $N \to \infty$, $h_i \to 0$, $i \in N$ in (8) we derive the results for the continual distributions of parameters across the thickness of the plate.

We do not write results for moments and resultants which are not complicated to receive. They do not completely define an internal stress state as opposed to the case of a single ply or a weakly nonhomogeneous plate. In comparison with Timoshenko type theories [$O(\varepsilon^4)$ represents absolute error] only the terms corresponding to the influence of the longitudinal shear appear in (8) but not of inertia of rotation.

The development of a 2-D approximate mathematical model of the contrast layered plate's dynamic bending under weak limits (6) is complete. The respective influences of stretching, shear and bending connect the equations.

The limits (6) are very wide and therefore our 2-D theory possesses some excessive complication as the price of generality. At the particular sets $p_i$, $r_i$, $q_i$ the equations are variously simplified. As a rule, the essential simplifications connect with the possibility of a separate description of flexion and the in-plane deformation. So, simpler models appear. We suggest a classification for them.

## 4. CLASSICAL CASE

First we consider plates with the asymptotically near parameters, when $p_i = q_i = 0$. The estimations $E_i/E_{max} \gg \varepsilon$, $c_{min}^2/c_i^2 \gg \varepsilon$ are practically enough for that. From (5)–(10) we obtain separate equations for $W$ and $V_\beta \equiv V_{\beta *} \equiv V^i_{\beta 0}$ ($V^i_\beta = V_\beta - zW_{,\beta}$ are complete displacements) and expressions for stresses, moments $M$ and resultants $Q$:

$$LW \equiv (D\Delta^2 + 2h\rho\partial^2/\partial t^2)W = q_z^- + \operatorname{div}[b\sigma_*^N + (2h-b)\sigma_*^1],$$

$$A_- \operatorname{div} \mathbf{V}_{,\beta} + A_+ \Delta V_\beta = -q_{\beta z}^- \quad (\beta = x, y), \quad V^i_\beta = V_\beta - zW_{,\beta}, \tag{11a}$$

$$\sigma_{xz} = \sigma^1_{xz*} - A_-(z) \operatorname{div} \mathbf{V}_{,x} - A_+(z)\Delta V_x + b_0(z)\Delta W_{,x} + O(\varepsilon) \quad (x \Leftrightarrow y),$$

$$\sigma_z = \sigma^1_{z*} + D(z)\Delta^2 W + \int_{z_1}^{z} \rho(x)\,dx\,W_{,tt} - b(z)\operatorname{div}\mathbf{q}^- - (z - z_1)\operatorname{div}\sigma^1_* + O(\varepsilon^2), \tag{11b}$$

$$M_x = -D(W_{,xx} + \nu W_{,yy}) + \alpha_* V_{y,y}, \quad M_{xy} = -D(1-\nu)W_{,xy} - \tfrac{1}{2}\alpha_* \operatorname{div}^* \mathbf{V},$$

$$Q_{xz} = D\Delta W_{,x} + 2h\sigma^1_{xz*} - A^0_- \operatorname{div} \mathbf{V}_{,x} - A^0_+ \Delta V_x; \quad \alpha_* = \int_{z_1}^{z_0} \alpha(z)\nu(z) z\,dz \quad (x \Leftrightarrow y), \tag{11c}$$

$$T_x = AV_{x,x} + A_0 V_{y,y} - \alpha_* W_{,yy} \, (x \Leftrightarrow y), \quad T_{xy} = \alpha_* W_{,xy} + \tfrac{1}{2}(A - A_0)\operatorname{div}^* \mathbf{V},$$

$$D(z) = \int_{z_1}^{z} \int_{y}^{z_0} \alpha(x) x\,dx\,dy, \quad 2A_\pm(z) = \int_{z_1}^{z} \frac{E(x)\,dx}{1 \pm \nu(x)} = \sum_{j=1}^{i-1} \frac{E_j h_j}{1 \pm \nu_j} + \frac{E_i(z - z_i)}{1 \pm \nu_i},$$

$$z_i \leqslant z \leqslant z_{i+1}, \quad D = D(z_0), \quad A_\pm = A_\pm(z_0), \quad A = \int_{z_1}^{z_0} \alpha(z)\,dz,$$

$$\rho = \int_{z_1}^{z_0} \rho(x)\,dx, \quad \nu = D^{-1} \int_{z_1}^{z_0} z^2 \nu(z)\alpha(z)\,dz,$$

$$b_0(z) = \int_{z_1}^{z} y\alpha(y)\,dy = \tfrac{1}{2}\alpha_i(z^2 - z_i^2) + \tfrac{1}{2}\sum_{j=1}^{i-1} \alpha_j(z_{j+1}^2 - z_j^2), \quad b_0(z_0) = 0,$$

$$Ab(z) = \int_{z_1}^{z} dy \int_{z_1}^{y} \alpha(x)\,dx = (z - z_i)[A^i + \tfrac{1}{2}\alpha_i(z + z_i)] + \sum_{j=1}^{i-1} h_j[A^j + \alpha_j z_{j+1/2}],$$

$$b = b(z_0), \quad A^j = \sum_{k=1}^{j-1} A_k - z_j\alpha_j, \quad A_k = \alpha_k h_k,$$

$$A_0 = \int_{z_1}^{z} \alpha(z)\nu(z)\,dz, \quad A^0_\pm = \int_{z_1}^{z} A_\pm(z)\,dz. \tag{11d}$$

Here the results are generalized for the case of the continual distribution of parameters through thickness. Asymptotic orders are identical to the case of homogeneous plate (Gol'denveiser, 1962). The Kirchhoff approximations take place and we can call (11) an improved classical theory of inhomogeneous plates. All the stresses are known, since, $\sigma'_x$, $\sigma'_y$ and $\sigma'_{xy}$ are defined by the usual formulae. Therefore the model describes a stretching–bending process of an equivalent homogeneous plate with average stiffness. Some differences are only apparent in (11c), (11d) because of the various Poisson's ratios. The stretching leads to the appearance of the moments and the tangential resultants depending on the transversal displacement $W$. The additional terms in (11d) are of the power $\varepsilon^{-1}$ although the other terms are O(1). They are small in the expressions for moments (11c).

The typical edge condition may be formulated in the case of a finite plate size. Their contribution makes an error O($\varepsilon$), generally speaking. We can assume that the improved conditions (Kolos, 1965) bear the solutions with a higher degree of accuracy than the classical ones as in the case of homogeneous plate.

## 5. EVALUATIONS OF FUNCTION ORDERS AND DISTURBANCES OF KIRCHHOFF'S HYPOTHESES

In this section local and mini-maximum asymptotic orders of unknown functions are evaluated. On this basis the restrictions for the indices $p_i$, $r_i$, ... are established when Kirchhoff's assumptions either remain in force or break in turn.

It is assumed that inertial forces have no more asymptotic power than the elastic strength in eqn (8a). Otherwise the short waves would appear in solutions as the spectrum-analysis shows. Then the comparison of asymptotic orders of terms in (8a), (8b) taking into account the limits (6) leads to the following evaluations for displacements and the characteristic time index $\gamma$:

$$W \sim \varepsilon^{-1-\kappa}, \quad V'_{\beta 0} \sim \varepsilon^{1-\kappa_{\beta}}, \quad V'_{\beta} \sim \varepsilon^{-2-\kappa}, \quad \kappa = \min(p_i + r_i + 2R_i),$$
$$S_i \geqslant \kappa_{\zeta} - r_i, \quad \gamma \leqslant \kappa_q - \kappa \leqslant 2, \quad \kappa_q = \min(p_i + r_i + q_i). \qquad (12)$$

Here $R_i$ denotes the asymptotic powers of the nondimensional distances from the "middle" plane (10) to the lamina number $i$:

$$\zeta = O(\varepsilon^{R_i}), \quad \zeta_i < \zeta < \zeta_{i+1}, \quad 0 \leqslant R_i \leqslant r_i,$$

a quantity $\kappa_{\beta i}$ will be determined below.

The deflection of orders from the common case is equal to $\kappa$: the plate consists of asymptotically weak layers only ($p_i + r_i \not\equiv 0, i \in I$).

Inertia is small at $\gamma < \kappa_q - \kappa$ and static bending takes place at $\gamma \leqslant \kappa_q - \kappa - 2$. In the case $\gamma = \kappa_q - \kappa$ inertia is asymptotically the same as the main terms in (8a) while short waves are still absent.

Now we write the sequences of terms of asymptotic orders (with opposite signs only) in a dimensionless view of eqns (8b,c), respectively,

$$\kappa_{\beta 1} - 2 - p_1 - r_1, \ldots, \quad \kappa_{\beta N} - 2 - p_N - r_N, 0, \qquad (13a)$$
$$\kappa_{\beta i}, \kappa_{\beta *}, \max_{k \in K_i}(p_k - \tilde{r}_k), \quad \varphi_i, \psi_i,$$

$$(\varphi_i, \psi_i) = \min_{m=1,N} \max_{j \in I, k \in K^m_{ij}} (\kappa_{kj} + \kappa_{\beta j}, \kappa_{kj} + \kappa - R_j + 3), \quad \kappa_{\beta *} \equiv \kappa_{\beta i *},$$

$$\kappa_{kj} = p_k - \tilde{r}_k - 2 - r_j - p_j, \quad \tilde{r}_k = r_k, \quad k \neq i, \quad \tilde{r}_k = R_k, \quad k = i,$$
$$K^m_{ij} = K_i \cap \{k : k > j, m = 1; k < j, m = N\},$$
$$K_i = \{k : k = i_* + 1, \ldots, i, i > i_* + 1; k = i_* - 1, \ldots, i, i < i_* - 1; \phi, i = i_*, i_* \pm 1\}. \qquad (13b)$$

A case $K^m_{ij} = \phi$ implies the elimination of respective terms in (8c). If all the sets $K^m_{ij} = \phi$ then indices $\varphi_i, \psi_i$ vanish from (13b) entirely. The double meaning $m = 1, N$ is the consequence of different function representations given in (8c). Explaining a physical sense of some terms in (12), (13). Summations $p_k+r_k$ and residuals $p_k-r_k$ characterize the asymptotic orders of lamina rigidities $A_k$ and $B_k$ with respect to stretching and transversal shear, respectively. The sum of $p_k+r_k+2R_k$ determines a power of contribution of the $k$th ply to the monolithic bending rigidity $D$.

As a rule, $\kappa_{kj} \leqslant 0$. So, this is true at $p \leqslant 2$. Then the requirement of consensus of asymptotic powers in eqns (8b) using (13b) leads to the equality

$$\kappa_{\beta i} = \max\{\kappa_{\beta *}, \max_{k \in K_i}(p_k-\tilde{r}_k), \varphi_i, \psi_i\}. \tag{14}$$

Substituting (14) in (13a) and taking into account the inequalities

$$p \leqslant 3, \quad \kappa_{kj} \leqslant 0, \quad \kappa_{kj}+\kappa_{\bar{i}}-3-p_i-r_i-R_i = p_k-\tilde{r}_k-1+\kappa-p_j-r_j-p_i-r_i-R_i < p_k-\tilde{r}_k-1,$$

in so far as

$$p_i+r_i+p_j+r_j+R \geqslant \min_{j \in I}(2p_j+2r_jR_j) \geqslant \kappa,$$

we find

$$\kappa_{\beta *} = \kappa_\beta \equiv 2+\min(p_i+r_i), \quad \kappa_{\beta i} \leqslant \kappa+3.$$

Suppose now that some quantities $\kappa_{kj}$ are positive. In view of (6) their maximum value is reached at $r_k = r_j = p_j = 0$ and equals unity. Then in eqn (8b) the last two sums contain the main summations corresponding to the value $\kappa_{kj} = 1$. Equalling the highest power terms in this equation we obtain $\kappa_{\beta j} = \kappa+3$ for some $j \in I$. For the remaining numbers $j$ we can prove $\kappa_{\beta j} \leqslant \kappa+3$. As a result, an important condition is proved

$$\kappa_\beta \leqslant \kappa_{\beta i} \lesssim \kappa+3, \tag{15}$$

and, at the same time, the second assumption (7).

Thus, all the hypotheses (7) have been proved. Simultaneously the condition $\psi_i \geqslant \varphi_i$ is just and after that (14) becomes the definition of $\kappa_{\beta i}$, indeed. The absolute errors in the formulae (8f) for $\sigma_z$, $\sigma_{\beta z}$, $\sigma_\beta$ ($\sigma_{xy}$ as $\sigma_\beta$) are $O(\varepsilon^2)$, $O(\varepsilon)$, $O(\varepsilon^{p_i-\kappa})$ and the components equal $O(\varepsilon^0)$, $O(\varepsilon^{-g_i})$, $O(\varepsilon^{p_i-\kappa-2})$, respectively, where

$$g_i = \max(0, 1+\kappa-G_i), \quad 0 \leqslant g_i \leqslant 1+\kappa,$$
$$G_i = \max\{\min_{1 \leqslant j \leqslant i}(p_j+\tilde{r}_j+R_j), \min_{N \geqslant j \geqslant i}(p_j+\tilde{r}_j+R_j)\}. \tag{16}$$

The rule (16) is complex because the main terms in some representations of (8d) can be eliminated owing to eqns (9). For example,

$$N = 2, \quad p_1 = r_2 = 0, \quad p_2 = p \geqslant r_2 = r, \quad R_i \equiv 0, \quad \kappa = \min(p, r) = r, \quad G_1 = 0,$$
$$G_2 = p, \quad g_1 = 1+r, \quad g_2 = 1+r-p, \quad r \leqslant p \leqslant 1+r; \quad g_2 = 0, \quad p \geqslant 1+r.$$

Note the index $q_i$ influences the choice of $\gamma$ and hence of the characteristic process time $t_0$ only. But after that we will have proved the quasi-statics of longitudinal movements of our plates.

*Mutual relation with Kirchhoff's assumptions*

Below we will compare: (a) increments of displacements $v_\beta^i$ at a jump from the $i$th on the $(i+1)$th interface corresponding to bending with similar increments from the transversal shear; (b) analogical residuals of the normal displacement $v_z^i$ with its piece-constant component $w_i$:

$$v_\beta^i(z_i)-v_\beta^i(z_{i+1}) = \varepsilon H_i\left[\frac{\partial w_i}{\partial\beta}-\frac{\tau_{\beta z0}^i}{\mu_i e_i\varepsilon^{p_i+1}}\right] = \varepsilon H_i\frac{\partial w_i}{\partial\beta}[1+O(\varepsilon^{2-m_i})],$$

$$v_z^i(z_{i+1}) = w_i\left[1+\frac{H_i\varepsilon^{-p_i}\tau_{z0}^i}{w_i e_i(\lambda_i+2\mu_i)}\right] = w_i[1+O(\varepsilon^{3-n_i})]. \quad (17)$$

In view of $\tau_{z0}^i = O(1)$ and the estimation (16) we conclude a rule of the calculations of quantities

$$m_i = p_i-\min(1+\kappa, G_i), \quad n_i = p_i-r_i-1-\kappa,$$

$$m_* = \max m_i = \max\{p-1-\kappa, \max(p_i-G_i)\}, \quad n_* = \max(p_i-r_i)-1-\kappa. \quad (18)$$

They characterize a local influence of the transversal deformations $\gamma_{\beta z}$ and $\varepsilon_z$ on the stress states of plates, as a whole. In this way we suggest a classification of solutions with dependence on deviations from Kirchhoff's assumptions.

(1) $m_* = 0, n_* \leqslant 0$. All these assumptions are true as follows from (17), (18). The nontrivial cases are described by the inequalities

$$\max(p_i-r_i) \leqslant 1+\kappa, \quad \max(p_i-r_i-G_i) \leqslant 0.$$

For example, the case $p_1 = r_1 > 0$, $p_2 = r_2 = 0$, $p_3 = r_3 \geqslant 0$, $N = 3$ satisfies these conditions.

*A lamina of a number i is called rigid* if Kirchhoff's hypotheses hold for it ($m_i = 0, n_i \leqslant 0$).

(2) $0 < m_* < 1, n_* \leqslant 1$. Intermediate case. There is however one lamina where the concept of the united rigid normal is not in force. But at $m_* < 1$ the correction to the rigid normal angle rotation is sufficiently small and consists of $o(\varepsilon)$ regarding the main term.

(3) $1 \leqslant m_* \leqslant 3, n_* \leqslant 1$. The mentioned correction ranges from $O(\varepsilon)$ to $O(\varepsilon^{-1})$ and has the same ($m_* = 2$) or greater ($m_* > 2$) influence than the bending regarding the in-plane displacements. The first hypothesis is false, but the concept of the individual rigid normal and the second hypothesis about undeformability of the normal are in force for the present.

*A lamina having indices $1 \leqslant m_i \leqslant 3$, $n_i \leqslant 1$ will be called soft*

(4) $2 \leqslant m_* \leqslant 3, 1 < n_* \leqslant 2$. We shall speak about *transverse-soft lamina* if $2 \leqslant m_i$, $1 < n_i$. The contribution of normal deformation $\varepsilon_z$ of such lamina must be added. Hence, the assumption about undeformability of the normal does not hold. But until $n_* \leqslant 2$ the mentioned contribution is not great and may be considered as a small correction. Note that the limiting value $n_* = 2$ is reached at $p_i = 3, r_i = 0$.

Consider the values $p > 3$ for a short period of time. The deformation of the normal is established to be comparable with the bending effect at $n_* = 3$ ($p \geqslant 4$) or asymptotically greater at $n_* > 3$. Let a transverse-soft lamina or a number $k$ dislocate between two rigid layers $k \pm 1$. They bend independently of each other (in the limits of permissible error) if normal stress in the $k$th ply evaluating according to the simple formula

$$\sigma_z^k \approx E_k h_k^{-1}(W^{k-1}-W^{k+1}) = \sigma_* e_k \varepsilon^{p_k-r_k} H_k^{-1}(w^{k-1}-w^{k+1})$$

does not have a negative order with respect to $\varepsilon$. Thus, the criterion for independent bending of layers separating by transversal soft lamina is

$$p_k \geqslant 4+r_k+\max(\kappa_{k-1}, \kappa_{k+1}) \geqslant 4+r_k, \quad (w^k = O(\varepsilon^{\kappa_k})). \quad (19)$$

It is also a sufficient condition, since the equality $\sigma_{\beta k}^k = O(1)$ then holds.

In conclusion of this section we recall that 2-D theories of plates and shells are developed under obvious or unobvious assumptions about the independence of loads on $\varepsilon$ (i.e. $\sigma_*^m = O(1)$). The interface tractions, as a rule, depend on $\varepsilon$ in the main. On this cause the superposition of separate equations for individual layers may be used for the construction of a united model of layered plates only under condition (19) for transverse-soft laminae between rigid layers.

## 6. CLASSIFICATION OF DEGENERATE INHOMOGENEOUS STRUCTURE MODELS

Besides absolute maximums of $p_i, q_i, r_i$ and their sums as in Section 4 the mutual place of various layers is important when considering the responses of the structure as a whole. We identify three classes of plates and, consequently, the models of behavior.

*First class*

It is noted that very soft external laminae have no influence on the bending of a plate, as a whole. It is possible to give a general description of stacking sequences when the main equations are the same as in the improved classical theory.

With the operator div acting on both sides of eqns (8a)–(8c) and using some substitutions we derive the following consequences:

$$D^m\Delta^3 W + LW = A'\Delta V_* + A_i b_i \Delta \operatorname{div} \sigma_*^m + \sum_{i\in I}\sum_{j\in I} A_i'\Gamma_{ij}^m \Delta^2 V_j + q_z^- + h \operatorname{div} \mathbf{q}^+, \quad (20a)$$

$$A' = \sum_{i\in I} A_i', \quad A\Delta V_* = A_i G_i^m \Delta^3 W - \operatorname{div} \mathbf{q}^- - b_i A_i \Delta \operatorname{div} \sigma_*^m - \sum_{i\in I}\sum_{j\in I} A_i \Gamma_{ij}^m \Delta V_j,$$

$$D^m = A_i' G_i^m, \quad V_* = \operatorname{div} \mathbf{V}_*. \quad (20b)$$

We receive $A_1^1 = D^m = 0$ at $N = 1$ and hence eqns (20a), (20b) are separated. At $N = 2$ all the additional crossing terms vanish in (8c). So the operator $\Delta V_*$ may be expressed through loads from (20b). The substitution of this result in (20a) taking into account the equality $D^m = 0$ enable us to reproduce the classical dynamic bending plate equations too. The stress states of soft laminae are defined algebraically through $V_{\alpha*}$. Similar equations are at the elimination of the limits (6) and/or at $N > 2$ and at the rigid internal laminae ($p_i = 0, i = 2, 3, \ldots N-1$). For determining all the similar variants we transform eqns (20) into the dimensionless form and evaluate summations containing the operators $\Delta V_j$ and $\Delta^3 W$. Only these terms are additional to the weak contrast case. Their orders equal $3+a_i-\varphi_i$, $3+a_i-\psi_i$ in (20a) and $2+p_i+r_i-\varphi_i$, $2+p_i+r_i-\psi_i$ in (20b), where

$$a_i = p_i + r_i + s_i \geq p_i + r_i, \quad \psi_i \geq \varphi_i, \quad \zeta_1 + \zeta_N - \zeta_i - \zeta_{i+1} = O(\varepsilon^{s_i}).$$

We introduce the quantities

$$\varphi = \min(1 + p_i + r_i - \varphi_i), \quad \psi = \min(1 + p_i + r_i - \psi_i), \quad (\psi \leq \varphi).$$

The requirement $\psi \geq 0$ is the sufficient and necessary condition for the elimination of all the above terms, of course, in the limits of the given accuracy. Indeed, then the governing terms in (20a), (20b) have powers $O(\varepsilon^0)$ and the additional terms will be $O(\varepsilon^2)$. This inequality is equivalent to

$$p_k - r_k \leq \min_{j\in J_*}(p_j + r_j) + \min_{j\in J_*}(p_j + r_j + R_j) - \kappa, \quad k \in I, k \neq i_*,$$

$$J_* = \{j\colon 1 \leq j < k < i_* \cup i_* < k < j \leq N\}. \quad (21)$$

Thus, the following statement is proved.

Table 1. The values of $p_i$ when the plates ($N = 3, 4$) belong to the first class

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | 0 | 1 | 2 | 3 | $p_1$ | 0 | 1 | | | 2, 3 | |
| $p_2$ | 0 | 0–2 | 0–3 | 0–3 | $p_2$ | 0 | 1 | 2 | 1 | 2 | 3 |
| | | | | | $p_3$ | 0 | 0–2 | 0–2 | 0–2 | 0–3 | 0–3 |

*Theorem. Let the dispositions of plies in a plate be satisfied by inequalities* (21). *Only then equations for the reference functions* $W$, $V_{\beta *}$ *become the same as in the classical case* (11a) *like the relations for stresses in the group of rigid plies with the reference ply* (11b). *In the soft laminae* ($p_i \neq 0$) *expressions* (8f) *take place, but often they become algebraic* (*or else part of them are to remain differential equations solved separately*) *after evaluations and omitting small terms in each concrete case.*

*Consequence 1.* The theorem is in force under a simpler sufficient condition

$$p_k - r_k \leqslant 2 \min_{i \in J_*} (p_i + r_i) - \kappa$$

which is close to (21), or under the condition of not increasing indices $p_i$ in going up and down from the reference ply.

*Consequence 2.* If $r_i \equiv 0$ the condition (21) leads to

$$p_k \leqslant 2 \min p_j, \quad k < j \leqslant N, \; k > i_*; \quad 1 \leqslant j < k, \quad k < i_*.$$

All the cases $N = 1, 2$ belong to the first class. In Table 1 the other cases are shown at $N = 3, 4$, $r_i \equiv 0$, $p_N = 0$, $i_* = N$.

Making stricter the influence of loading and inertia terms it is possible to calculate the stresses in the soft external laminae with a higher relative accuracy than that deduced from the above formulae (8f). This influence increases as Young's moduli decrease for external layers and becomes comparable with or dominates over the bending response. It only matters when edge conditions on the cross-section do not play the main role. For example, it is desirable to determine more perfectly the stress field under a punch acting on the very soft lamina. Here we only qualitatively analyse the influence of different deformation processes on the solution with increasing external layer softening.

Let us consider plates with rigid internal laminae and soft external ones. In accordance with the theorem the stress state of rigid laminae is described by expressions (11). Incorporating these plies into unit rigid layers we will characterize its thickness by an index $r \geqslant 0$, its dynamic feature by a value $q \geqslant 0$ and calculate $p = \max(p_1, p_N)$, $\kappa = \min(p, 3r)$. We also propose $r_1 = r_N = 0$ and $R = r$ that means the center of bending lies in a rigid lamina.

First of all the improved classical theory takes place with respect to all the laminae at $0 \leqslant p \leqslant r$ and the bending stresses are significant at $0 \leqslant p \leqslant 3r$. At $p > r$ the displacements in the external laminae are expressed algebraically only through displacements of the rigid internal lamina and the loads and the asymptotic power of bending stresses in soft layers falls. So, at $p = 3r+1$, $p = 3r+2$ and $p = 3r+3$ they are $O(\varepsilon^{-1})$, $O(\varepsilon^0)$ and $O(\varepsilon^1)$, respectively and become comparable with the direct influence of loads and inertia. If $p = 3r+2+n, n > 0$ then the first $n$ orders of stresses are determined as at the dynamic deformation of a thin elastic layer on the rigid foundation.

We complete the research of the first class plates $\psi \geqslant 0$, $\varphi \geqslant 0$.

Table 2. The cases of condition (21) being broken ($N = 3$, $p_3 = 0$, $r_i = 0$)

| | | | | |
|---|---|---|---|---|
| $p_1, p_2$ | 0, 1 | 0, 2 | 0, 3 | 1, 3 |
| $\psi$ | $-1$ | $-2$ | $-3$ | $-1$ |
| $\varphi$ | 0 | $-1$ | $-3$ | 0 |
| $\kappa_{\beta i}$ | 2 | 2 | 3 | 2 |
| $m_*$ | 1 | 2 | 3 | 2 |
| $n_*$ | 0 | 1 | 2 | 2 |

*Second class*

It is defined by two inequalities $\psi < 0$, $\varphi \geqslant 0$. In Table 2 the cases of condition (21) being broken and the consequent values $\varphi$, $\psi$, $\kappa_{\beta i}$, $m_*$, $n_*$ are presented.

We see that variants $\psi < 0$, $\varphi \geqslant 0$ are possible and the second class is not empty. Then terms with $\Delta V_i$ can be neglected and with $\Delta^3 W$ must be kept in (20). The main equations separate on a sixth order one for $W$ and are usually the same with perturbed right-hand side for $V_{\beta *}$:

$$(D_0\Delta^3 + L)W = q_z^- + \operatorname{div}(h\mathbf{q}^+ - A'A^{-1}\mathbf{q}^-),$$

$$(A_i A_{i\beta})V_{\beta *} = -q_{\beta z}^- + (A_i G_i^m)\Delta^2 W_{,\beta}, \quad D_0 = (A_i' G_i^m) - (A_i G_i^m)A'/A,$$

$$V_\beta^i = V_{\beta *} + \left\{\frac{z}{\mu_i}\sum_{j\in J_m} e_j A_j \operatorname{sgn}(i-j) - G_i^m\right\}\Delta W_{,\beta} - zW_{,\beta} + O(\varepsilon^2 V_\beta^i).$$

The relations for stresses (8f) also permit some simplifications in each concrete case. Although the equations are separated the problem remains connected, as a whole, because of the lateral boundary conditions. The second class is physically characterized by the existence of several groups of rigid plies in the plate connected by the not very soft laminae.

In some cases eqns (8) without the second and third terms in the right-hand side of (8c) seem to be utilized more conveniently. We suggest solving them by a method of successive approximations because of the small size of the additional term in (8a).

*Third class*: $\varphi < 0$

Physically the third class is formed by the plates with some very soft intermediate laminae. Equations (8) remain connected and have the highest combined order but, in particular, they can be simplified. We show this in an example of plates with the alternate soft and rigid laminae.

## 7. PARTICULAR CASE OF PLATES WITIH ALTERNATE LAMINAE

Let us consider the plates characterized by the following alternate sets of parameters:

$$p_i = 0, \quad r_i = r \geqslant 0, \quad i = 1, 3, \ldots, N; \quad p_i = p > 0, \quad r_i = 0, \quad i = 2, 4, \ldots, N-1,$$

i.e. consist of alternate layers: (1) rigid and may be very thin; (2) soft and relatively thick. We suggest classification based on indices $p$ and $r$.

(1) $0 < p \leqslant r$. There are no considerable deflections from the theory (13)–(15), the model belongs to the first class.

(2) $r < p \leqslant 1 + r$, $-1 \leqslant \psi < 0$, $\varphi \geqslant 0$. We deal with a case from the second class. The equations take the form

$$(D_0\Delta^3 + L)W = q_z^- + \operatorname{div}(hq^+ - A'A^{-1}q^-),$$

$$A_-\operatorname{div} V_{,\beta} + A_+\Delta V_\beta = -q_{\beta z}^- + (A_i G_i^0)\Delta W_{,\beta} \quad (V_\beta \equiv V_{\beta 0}^1),$$

$$V_\beta^i = V_\beta + [G_i^0 + \gamma_i(z)]\Delta W_{,\beta} - zW_{,\beta} + O(\varepsilon^2 V_\beta^i),$$

$$G_i^0 = \sum_{j=1,3,\ldots}^{i-2} z_{j+1/2} A_j \sum_{k=j+1, j+3,\ldots}^{i-1} B_k^{-1} - \gamma_i(z_i), \quad \gamma_i = 0, \quad i = 2n-1,$$

$$\gamma_i = \frac{z}{\mu_i}\sum_{j=1,3,\ldots}^{i-1} A_j z_{j+1/2}, \quad i = 2n, \quad D_0 = A'(G_i^0 A_i)/A - (G_i^0 A_i').$$

The relations for stresses (8d) are not rewritten here. This variant has not been involved in Bolotin's theory for the similar structures, based on the Kirchhoff and Timoshenko approximations, where the energy of bending (transversal) deformations has been omitted in soft (rigid) laminae. The estimations of these parts of energy show that they are comparable with complete elastic energy in the soft layers and hence none is neglected.

(3) $1+r < p \leqslant 3$. Third class. We make the substitution

$$U^i_\beta = V^i_{\beta 0} - z_{i+1/2} W_{,\beta},$$

in the main equations and after some transformations obtain a new system of equations containing the lower order derivations of the function $W$

$$D_s \Delta^2 W + 2h\rho W_{,tt} + \sum_{i=3,5,\ldots}^{N} B_{i-1} d_{i-1} [U_i - U_{i-2} - d_{i-1} \Delta W] = q_z^- + \tfrac{1}{2}\operatorname{div}(h_1 \boldsymbol{\sigma}^1_* + h_N \boldsymbol{\sigma}^N_*),$$

$$A_i \Lambda_{i,\beta} \mathbf{U}^i + B_{i+1}(U^{i+2}_\beta - U^i_\beta - d_{i+1} W_{,\beta}) - B_{i-1}(U^i_\beta - U^{i-2}_\beta - d_{i-1} W_{,\beta}) = 0,$$

$$A_m \Lambda_{m\beta} \mathbf{U}^m + B_{m\pm 1}(U^{m\pm 2}_\beta - U^m_\beta - d_{m\pm 1} W_{,\beta}) = \mp \sigma^m_{\beta z *}, \quad m = 1, N; \quad i = 3, 5, \ldots, N-2,$$

$$D_s = \sum_{i \in I} E_i h_i^3 / 12(1-\nu_i^2), \quad U_i = \operatorname{div} \mathbf{U}^i d_i = \tfrac{1}{2}(h_{i-1} + 2h_i + h_{i+1}).$$

Here at $m = 1$ ($m = N$) we have a high (low) sign. These equations are close to those by Bolotin and Novitchkov (1980) where $r = 0$. There are some small corrections of $O(\varepsilon)$ in the first equation. We can also conclude that the deformations are formed according to Timoshenko's hypothesis in the soft laminae and they can be simulated by Winkler's shear type plies.

## 8. A FEW WORDS ABOUT LATERAL CONDITIONS FOR FINITE SIZE PLATES

The development of asymptotically true different edge conditions is a very complex and cumbersome problem. Therefore, until we can suggest some recommendations proceeding from common sense and an engineering approach it is only possible to use the classical conditions of free edge or rigid fixation or others. They give a zero approximation at the homogeneity. But it is necessary to add the following corrections. The main principle is to put tangential loads to lateral cross-section of every group of rigid laminae separately if there are internal soft layers in a plate. And all the lateral loads must be balanced so that only the self-balanced part of them remains on each lateral cross-section of the mentioned group. First, the number of conditions will correspond to the order of the system of equations. Second, weak damping and nondamping boundary layers have been revealed by Gussein-Zade (1968) and at $p \leqslant 2$ they must become components of the internal main integral.

We can explain the physical reason for the boundary effects as an example. Consider a spring consisting of some elastic plies with slipping contact. Evidently it will be dispersed under general self-balanced loads on its whole lateral cross-section. The same reason gives rise to the penetrating boundary layers in the structures with soft laminae. These conjectures need a proof, of course.

Some of the edge conditions mentioned have been deduced by means of the variation method by Bolotin and Novitchkov (1980). Some boundary layers were studied there too.

It is difficult to give advice about how to deduce improved conditions for this type of high contrast plate with the necessary research of boundary layer problems like the case of homogeneity (Kolos, 1965), where they give $O(\varepsilon)$ approximation corresponding to the accuracy of the equations. The causes are as follows: they will be very cumbersome for practical use. On the other hand, the equation's precision is often more valuable for the internal solution (which is not discussed further here) than the edge condition accuracy because: (a) mathematically speaking, the latter acts as an operator on a set with little measure in contrast with the first; (b) as a rule, the conditions of the real fixation of a plate itself are often modelled with a large error, especially for high contrast plates; (c) they only form their boundary layers in a short wave solution. As a consequence, the first known term of Weyl's asymptotic for the number of eigenvalues for problems of shell vibrations do not depend on the type of lateral conditions. Therefore, from a practical point of view, the equations seem advisably to use a higher degree approximation than for the edge conditions for the balance of the general accuracy of an internal problem solution. This is especially advisable if it will avoid additional difficulties.

Besides, the common conditions hold near to O(ε) approximations when the solution does not depend on the contour coordinate so the differences in classical and improved conditions are practically eliminated. The important class of axisymmetric and plane problems belongs to this case. Lastly, in the research of local effects far from the lateral edge, when the conditions on this edge play minor roles, they can be stated more approximately.

Nevertheless, the strict asymptotic research of edge effects is useful for the verification of classical theoy assumptions. In particular, the establishment of conditions of boundary layer damping as well as that of the small corrections to classical conditions is of great interest.

## 9. CONCLUSIONS

The 2-D mathematical models for the dynamics of thin, high, nonhomogeneous elastic plates have been presented. In general, bending and stretching are strongly connected as in the Timoshenko–Reissner theory of plates. Simple modifications have been separated and classified. In particular, the improved classical theory of layered plates has been followed. Then bending and in-plane motions are separately described in spite of dynamics and differences in Poisson's ratios. That is a substantiation of the engineering approaches. But all the stress components have been determined in the present way.

The suggested classification of equations depends on mutual lamina locations and the contrast of layers with regard to Young's moduli and thicknesses. For the separated first class of plates the theory turns out to be no more complicated than the classical one. These are plates with a rigid core of layers inside and weaker laminae at the periphery. For example, any two-layered plate belongs to the first class. In the second intermediate class (there are internal weak laminae) the more complex sixth order dynamic equation for bending and the usual ones for stretching take place. As to the third class characterized by the presence of a high weak internal lamina the general equations have been simplified in the case of alternative layers. Lastly the true edge condition statements for finite size plates have been discussed.

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